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1990 J. Phys. A: Math. Gen. 23 L511

(http://iopscience.iop.org/0305-4470/23/10/007)

J. Phys. A: Math. Gen. 23 (1990) L511-L515. Printed in the UK

LETTER TO THE EDITOR

A soluble seven-vertex model for clusters with interfacial bending energy

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Received 12 March 1990

Abstract. A model describing an ensemble of disjoint clusters on a square lattice is introduced. In addition to the usual surface energy terms, ε_x and ε_y, the model includes a bending energy ε_c per corner in the interface. It is formulated as a seven-vertex model which is in general not solvable. For $\beta\varepsilon_c = \ln(2)/2$, however, this seven-vertex model becomes a free fermionic one and an exact expression for its free energy is derived. In the extreme anisotropic ('Hamiltonian') limit the operator content coincides with that of the 1D Ising model in a transverse field. Three critical lines with Ising-like behaviour are found.

In many physical systems which undergo phase separation at low temperature, the energy has not only a term proportional to the interfacial area but also a contribution due to the extrinsic curvature of the interface [1]. For instance, a flat interface usually has lower energy with additional work being required to bend the surface while maintaining the surface area. Typical examples include micellar systems, microemulsions and other exotic fluids [2]. Much effort has been focused recently on understanding the effect of such curvature terms on different models of random surfaces. Their classification, according to the exact form of the interactions, is still vigorously investigated [3, 4].

Two-dimensional (2D) systems with stiffness have also been studied [5-8]. Finite stiffness is not expected to change the long-range interfacial scaling properties since the persistence length l_p is finite and diverges only as the stiffness grows indefinitely large. If the cost to bend the interface by a right angle is ε_c; then on a scale l such that $\beta^{-1}\ln[l] > \varepsilon_c$, the entropy gained by bending the interface will overcome the energy cost and $l_p \approx \exp\{\beta\varepsilon_c\}$.

Although universality arguments suggest that the bulk properties of an ensemble of clusters near the critical temperature will follow a behaviour similar to that in the absence of the curvature term, it remains desirable to have an exactly solvable model which vindicates this conjecture. Ever since Onsager's exact solution for the 2D Ising model [9], many other 2D models have been solved. Notable among them are the six- and eight-vertex models [10-12]. Complete solutions for these can be obtained if they obey the so-called 'free fermion' condition [12]. These vertex models are generally viewed as ferroelectrics, though other interpretations are possible. To the best of our knowledge, however, any connection between cluster models and bending energy has not been considered previously.

0305-4470/90/100511+05$03.50

In the present letter we introduce a square lattice model for disjoint clusters with a contribution to the energy due to bending. This model has the structure of a seven-vertex model. For a given value of the bending energy, this seven-vertex model has a free-fermion representation which allows its complete solution with the results described below. (A seven-vertex model of a different sort has been considered in connection with a O(n) model on a hexagonal lattice [13].)

We begin with a description of the model. Consider a collection of disjoint clusters (boundaries of the clusters cannot self-intersect or touch each other) as depicted in figure 1. The partition function is given by:

$$Z(h, v, c) = \sum C(n_v, n_h, n_c) h^{n_h} v^{n_v} c^{n_c} \tag{1}$$

where $C(n_v, n_h, n_c)$ is the total number of configurations with n_v vertical bonds, n_h horizontal bonds and n_c corners. The Boltzmann factors are $h = \exp\{-\beta\varepsilon_x\}$, $v = \exp\{-\beta\varepsilon_y\}$, and $c = \exp\{-\beta\varepsilon_c\}$; where ε_x and ε_y are the energy per unit length of interface in the x and y directions respectively and ε_c is the energy per corner. The mapping of this model onto a seven-vertex model is depicted in figure 2 with the following weights (Boltzmann factors):

$$\omega(1) = \omega(2) = \omega(3) = \omega(4) = (hv)^{1/2} c \qquad \omega(5) = h \qquad \omega(6) = v \qquad \omega(7) = 1.$$

This partition function may be related to that of an Ising model with spins $s_i = \pm 1$ sitting at the centre of the plaquettes (sites of the dual lattice) such that the configurations like figure 1 correspond to its low temperature expansion [10]. The Hamiltonian will require interactions between nearest-neighbour (horizontal and vertical) and next-nearest-neighbour (diagonal) as well as four-spin interactions around a face. Such a Hamiltonian is

$$H_{\text{Ising}} = -\sum_j [J_0 + J_x s_j s_{j+e_x} + J_y s_j s_{j+e_y} + J_d (s_j s_{j+e_x+e_y} + s_j s_{j+e_x-e_y}) + J_4 s_j s_{j+e_x} s_{j+e_y} s_{j+e_x+e_y}] \tag{2a}$$

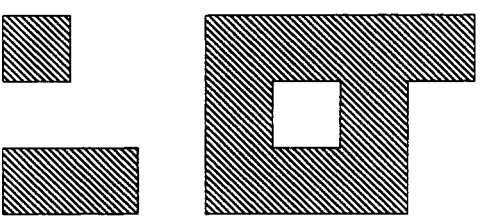

Figure 1. A typical configuration of disjoint clusters contributing to the partition function.

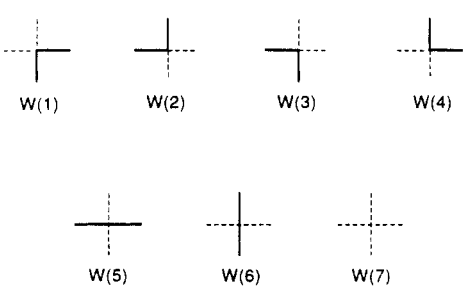

Figure 2. The seven vertices. Their relative weights $\omega(i)$ are defined in the text.

which typically generates an eight-vertex model. In order to eliminate the extra vertex and produce the rest with the appropriate weights, the couplings J_x, $J_y \to +\infty$ and J_0, J_d, $J_4 \to -\infty$ such that:

$$\varepsilon_x = J_0 + J_x - J_y - 2J_d + J_4 \qquad \varepsilon_y = J_0 - J_x + J_y - 2J_d + J_4$$
$$\varepsilon_c = J_0 - J_4 \tag{2b}$$

under the constraint that

$$J_0 + J_x + J_y + 2J_d + J_4 = 0. \tag{2c}$$

This description naturally yields the order parameter $m = \langle s_i \rangle$ when ε_x, ε_y, and ε_c are positive; m will be non-zero in the ordered phase which corresponds to small clusters and will vanish at the transition where the cluster size diverges. When $\varepsilon_x < 0 < \varepsilon_y$, the order parameter corresponds to alternating rows of up and down spin, i.e., $(k_x, k_y) = (0, \pi)$. Alternating columns appear when $\varepsilon_y < 0 < \varepsilon_x$. The inability of interfaces to intersect prevents the antiferromagnetic order parameter (i.e., $(k_x, k_y) = (\pi, \pi)$) from ever arising. In cases when ε_c is negative, that is when it is favourable to bend, other order parameters will occur.

We have found that for the special value of the bending term, $\beta\varepsilon_c = (\ln 2)/2$, i.e., $c = 1/\sqrt{2}$, the Boltzmann weights obey the general free fermion condition [12]. These diagrams have a representation in terms of two Grassmann (anticommuting) variables [14] per site, $\eta_{i,j}$ and $\eta^*_{i,j}$, with the following quadratic (and hence soluble) Hamiltonian

$$H_G = \sum_{i,j} \eta^*_{i,j}\eta_{i,j} - h\eta^*_{i,j}\eta_{i+1,j} + \frac{v}{2}[-\eta^*_{i,j}\eta_{i,j+1} + \eta_{i,j}\eta^*_{i,j+1} - \eta^*_{i,j}\eta^*_{i,j+1} + \eta_{i,j}\eta_{i,j+1}]. \tag{3a}$$

Note that it is the Boltzmann weights which appear in the Grassmann 'Hamiltonian' H_G. The partition function (1) consists of the integration over these anticommuting variables

$$Z\left(h, v, c = \frac{1}{\sqrt{2}}\right) = \int \prod_{i,j} \mathrm{d}\eta_{i,j}\, \mathrm{d}\eta^*_{i,j} \exp\{H_G\}. \tag{3b}$$

Consider the system on an $L \times L$ lattice with periodic boundary conditions; the Fourier transformation diagonalises the Hamiltonian. Allowing L to approach infinity and solving for the free energy per site f we find:

$$\beta f = -\frac{1}{8\pi^2}\int_{-\pi}^{\pi} \mathrm{d}k_x \int_{-\pi}^{\pi} \mathrm{d}k_y \ln(1 + v^2 + h^2 - 2v\cos k_y - 2h\cos k_x + 2hv\cos k_x \cos k_y) \tag{4}$$

From an expansion of the argument of the logarithm in small momenta, we expect a phase transition when $v + h = 1$. Defining $t = 1 - h - v$ to measure the distance from the critical point, we find a $t^2 \ln(t^2)$ behaviour for the singular part of the free energy, which is typical of an Ising-like transition. Equation (4) also implies that the correlation length diverges as t^{-1}. Because the vs always appear in pairs in the partition function, there is a symmetry between v and $-v$ (similarly for h). Consequently, we also have critical lines at $v - h = 1$ and $-v + h = 1$ (see figure 3). The transitions across these lines are also in the Ising universality class.

We can make the connection between this model and the Ising model in the following way. Recalling that the partition function of a 1D fermionic Hamiltonian has a 2D path integral formulation in terms of Grassmann variables [15], we can find

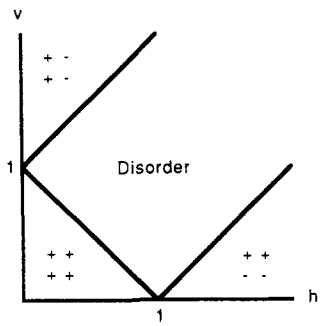

Figure 3. The phase diagram in the $v-h$ plane for $c=\exp\{-\beta\varepsilon_c\}=1/\sqrt{2}$; all transitions are Ising-like.

the 1D quantum Hamiltonian which 'corresponds' to this seven-vertex model. We start with the Trotter product formula

$$Z=\operatorname{Tr} e^{-\beta'\hat{H}}=\operatorname{Tr}(e^{-(\beta'/L)\hat{H}})^L=\operatorname{Tr}\hat{T}^L \tag{5}$$

where $\hat{H}$ is a 1D fermionic Hamiltonian. Employing the following definition of and relations among the coherent states,

$$|\eta_{i,j}\rangle=\exp\{ac_j^+\eta_{i,j}\}|0\rangle \tag{6a}$$

$$c_l|\eta_{i,j}\rangle=a\eta_{i,j}\delta_{j,l}|\eta_{i,j}\rangle \tag{6b}$$

$$\langle\eta_{i,j}|\eta_{k,l}\rangle=\exp\{a^2\eta_{i,j}^*\eta_{k,l}\delta_{j,l}\} \tag{6c}$$

$$\hat{1}=\prod_j\int\frac{d\eta_{i,j}^*\,d\eta_{i,j}}{a^2}\exp\{-a^2\eta_{i,j}^*\eta_{i,j}\}|\eta_{i,j}\rangle\langle\eta_{i,j}| \tag{6d}$$

where c are fermionic operators, η are Grassmann variables and a is some real number, one can write

$$\begin{aligned}Z=a^{-2L}\int d\eta_{1,j}^*\,d\eta_{1,j}\exp\{-a^2\eta_{1,j}^*\eta_{1,j}+a^2\eta_{1,j}^*\eta_{2,j}\}T(\eta_{1,j}^*,\eta_{2,j})\\ \times d\eta_{2,j}^*\,d\eta_{2,j}\exp\{-a^2\eta_{2,j}^*\eta_{2,j}+a^2\eta_{2,j}^*\eta_{3,j}\}T(\eta_{2,j}^*,\eta_{3,j})\\ \times d\eta_{L,j}^*\,d\eta_{L,j}\exp\{-a^2\eta_{L,j}^*\eta_{L,j}+a^2\eta_{L,j}^*\eta_{1,j}\}T(\eta_{L,j}^*,\eta_{1,j}).\end{aligned} \tag{7}$$

Equations (7b) and (7c) lead to $\langle\eta_{i,j}|\hat{T}|\eta_{i+1,j}\rangle=\exp\{a^2\eta_{i,j}^*\eta_{i+1,j}\}T(\eta_{i,j}^*,\eta_{i+1,j})$ where $T(\eta_{i,j}^*,\eta_{i+1,j})$ is $\hat{T}$ with c_j^+ replaced by $\eta_{i,j}^*$ and c_j by $\eta_{i+1,j}$.

Let $h\to -h$ in H_G and compare $\exp\{H_G\}$ and T. Then supposing h is large and $\eta_{i+1,j}-\eta_{i,j}=O(1/h)$, a little more manipulation leads to the following fermionic Hamiltonian:

$$\frac{\beta'\hat{H}}{L}=\sum_j\frac{v}{2h}[c_j^+c_{j+1}-c_jc_{j+1}^++c_j^+c_{j+1}^+-c_jc_{j+1}]+\frac{h+1}{h}c_j^+c_j \tag{8}$$

to first order in $1/h$.

A quantum spin Hamiltonian may be obtained by an inverse Jordan-Wigner transformation:

$$c_k=\exp\left[\pi i\sum_{j=1}^{k-1}\sigma_j^+\sigma_j^-\right]\sigma_k^- \tag{9}$$

where σ are the spin raising and lowering operators. Variables from the same site anticommute; while those from different sites commute. This substitution, along with flipping the sign of h back ($h \to -h$), yields

$$-\frac{\beta'\hat{H}}{L} = \sum_j \frac{v}{2h}[\sigma_j^+\sigma_{j+1} + \sigma_j\sigma_{j+1}^+ + \sigma_j^+\sigma_{j+1}^+ + \sigma_j\sigma_{j+1}] + \left(\frac{1-h}{h}\right)\sigma_j^+\sigma_j \qquad (10a)$$

Using the following representation

$$\sigma^z = \sigma^+ + \sigma^- \qquad \text{and} \qquad \sigma^x = 2\sigma^+\sigma^- - 1$$

leads to

$$-\frac{\beta'\hat{H}}{L} = \sum_j \left[\frac{v}{2h}\sigma_j^z\sigma_{j+1}^z + \left(\frac{1-h}{2h}\right)(\sigma_j^x + 1)\right]. \qquad (10b)$$

This set of transformations suggests that the original model in terms of Grassmann variables might be viewed in this $h \to \infty$ limit as the path integral formulation of the 1D quantum Ising model in a transverse field [16]. Utilising this equivalence, one can confirm the critical lines derived earlier and find the order parameter critical exponent β

$$\begin{aligned} &\langle s_i \rangle \sim \langle \sigma_i^z \rangle \sim t^{1/8} \qquad \text{when } v + h < 1 \\ &\langle s_i \rangle = 0 \qquad \text{when } v + h > 1. \end{aligned} \qquad (11)$$

To conclude, we have presented a model for disjoint clusters with both the usual surface energy terms and a term for bending. For a special value of the energy per corner, an exact solution was found; the transitions were determined to be Ising-like. Thus, for the first time a rigorous solution for a model with an interface bending energy has been presented. Universality arguments would suggest that the critical behaviour of the model for different values of the bending energy would remain in the Ising universality class.

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